KLEINIAN SINGULARITIES, DERIVED CATEGORIES AND HALL ALGEBRAS

M. Kapranov and E. Vasserot

The Kleinian singularities $\mathbf{C}^2/G$ associated to finite subgroups $G \subset SL_2(\mathbf{C})$, are of fundamental importance in algebraic geometry, singularity theory and other branches of mathematics. Despite the very classical nature of the subject, new remarkable properties continue to be discovered. One such discovery was the McKay correspondence [9] and its interpretation by Gonzalez-Springberg and Verdier [3] in terms of the minimal resolution $\mathbf{C}^2//G$. Their results give identifications

$$K_0(\mathbf{C}^2//G) \quad \simeq \quad \mathrm{Rep}(G) \quad \simeq \quad \widehat{\mathbf{h}}_{\mathbf{Z}},$$

where K_0 is the Grothendieck group, Rep is the representation ring and $\widehat{\mathbf{h}}_{\mathbf{Z}}$ is the root lattice of the affine Lie algebra (of type A-D-E) associated to G.

Our first goal in this paper is to extend the above results by describing the derived category of coherent sheaves on $\mathbf{C}^2//G$, instead of just K_0. Theorem 1.4 identifies it with the derived category of G-equivariant $\mathbf{C}[x,y]$-modules, i.e., of modules over the crossed product algebra $\mathbf{C}[x,y][G]$. It is surprising that such a basic fact has not been noticed before. Our approach can be seen as a refinement, in a purely algebraic setting, of the techniques of Kronheimer and Nakajima [7] in that we get rid of Dolbeault complexes with growth conditions at infinity, stability conditions for vector bundles and so on.

We then define $\mathbf{H}$, an Euler-characteristic version of the Hall algebra [12] of the category of coherent sheaves on $\mathbf{C}^2//G$ and apply the constructed equivalence to exhibit a subalgebra in $\mathbf{H}$ isomorphic to $U(\mathbf{g}_G^+)$. Here $\mathbf{g}_G^+$ is the nilpotent part of the finite-dimensional Lie algebra (of type A-D-E) corresponding to G. As a consequence, we get a result about any algebraic surface S equipped with a configuration $C = \bigcup_i \mathbf{P}_i^1$ of (-2)-curves intersecting transversally. Namely, taking the intersection graph of the $\mathbf{P}_i^1$ as a Dynkin graph, we get a possibly infinite-dimensional Kac-Moody Lie algebra, and the theorem is that the positive part of this algebra acts in the space of functions on isomorphism classes of coherent sheaves in S. This partly extends the results of Nakajima [10] to a wider geometric context.

§1. Equivalence of derived categories.

1.1. For a smooth algebraic variety X and an integer $m \geq 0$ we denote by $X^{(m)}$ the mth symmetric power of X and by $X^{[m]}$ the Hilbert scheme parametrizing 0-dimensional subschemes $\xi \subset X$ of length m. Given such a ξ, the corresponding point of $X^{[m]}$ is denoted $[\xi]$. We denote by $\mathcal{E}$ the tautological m-dimensional bundle on $X^{[m]}$, whose fiber at $[\xi]$ is $H^0(\xi, \mathcal{O})$. Let G be a finite group acting on X. Let $m = |G|$. The quotient X/G can be viewed as a closed subvariety in $X^{(m)}$. Suppose that the action is free on an open G-invariant set $U \subset X$. Define the Hilbert quotient $X//G$ as the closure of U/G in $X^{[m]}$, cf. [6]. The Chow morphism $X^{[m]} \to X^{(m)}$ gives a map $p : X//G \to X/G$. Let $\Sigma \subset (X//G) \times X$ be the incidence subscheme. Let $p_1 : (X//G) \times X \to X//G$, $p_2 : (X//G) \times X \to X$, be the projections and q_1, q_2 be the restrictions of p_1, p_2 to Σ. The restriction of the tautological bundle to $X//G$ is denoted again by $\mathcal{E}$; it is a bundle of G-modules isomorphic to the regular representation. If ρ, V are representations of G, with ρ irreducible, we set $V_\rho = \mathrm{Hom}_G(\rho, V)$.

1.2. Let $\mathcal{C}oh_G(X)$ be the category of G-equivariant coherent sheaves on X, and $\mathcal{C}oh(X//G)$ be the category of coherent sheaves on $X//G$. Define two functors $\Phi : D^b(\mathcal{C}oh_G(X)) \to D^b(\mathcal{C}oh(X//G))$ and $\Psi : D^b(\mathcal{C}oh(X//G)) \to D^b(\mathcal{C}oh_G(X))$ by

$$\Phi(\mathcal{F}) = (Rq_{1*}Lq_2^*\mathcal{F})^G = (Rp_{1*}(p_2^*\mathcal{F} \otimes^L \mathcal{O}_\Sigma))^G \quad \text{and} \quad \Psi(\mathcal{G}) = Rp_{2*}R\mathcal{H}om(\mathcal{O}_\Sigma, p_1^*\mathcal{G}).$$

The functors Ψ and Φ are adjoint, i.e. $\mathcal{H}om(\Phi(\mathcal{F}), \mathcal{G}) \simeq \mathcal{H}om(\mathcal{F}, \Psi(\mathcal{G}))$.

1.3. Let Z, W be algebraic varieties, equipped with actions of finite groups G, H respectively. Let p_W, p_Z be the projections from $Z \times W$ to W, Z respectively. Let $\mathcal{L}$ be an object of $D^b(\mathcal{C}oh_{G \times H}(Z \times W))$, such that each of the cohomology sheaves $\mathcal{H}^i(\mathcal{L})$ has proper support with respect to p_W. Taking $\mathcal{L}$ as a "kernel" defines a functor

$$F_\mathcal{L} : D^b(\mathcal{C}oh_G(Z)) \to D^b(\mathcal{C}oh_H(W)), \quad \mathcal{F} \to (Rp_{W*}(p_Z^*\mathcal{F} \otimes^L \mathcal{L}))^G.$$

If Z, W, T are varieties with actions of groups G, H, K respectively, and $\mathcal{L} \in D^b(\mathcal{C}oh_{G \times H}(Z \times W))$, $\mathcal{M} \in D^b(\mathcal{C}oh_{H \times K}(W \times T))$, the composition $F_\mathcal{M} \circ F_\mathcal{L}$ is isomorphic to $F_{\mathcal{M}*\mathcal{L}}$ where

$$\mathcal{M} * \mathcal{L} = (Rp_{13*}(p_{12}^*\mathcal{L} \otimes^L p_{23}^*\mathcal{M}))^H,$$

and the p_{ij} are the projections of $Z \times W \times T$ onto the pairwise products. In the case (1.2) of the product $(X//G) \times X \times (X//G)$ let $\Sigma_{12} = p_{12}^{-1}(\Sigma)$ and $\Sigma_{23}^t = p_{23}^{-1}(\Sigma^t)$, where $\Sigma^t \subset X \times (X//G)$ is the transpose variety of Σ.

Proposition. *In the situation of (1.2) the following is true:*
(a) The composition $\Phi\Psi$ has the kernel

$$\mathcal{L} = (Rp_{13*}(R\mathcal{H}om(\mathcal{O}_{\Sigma_{12}}, \mathcal{O}_{\Sigma_{23}^t})))^G \in D^b(\mathcal{C}oh(X//G \times X//G)).$$

(b) The composition $\Psi\Phi$ has the kernel

$$\mathcal{M} = Rs_{13*}(R\mathcal{H}om(\mathcal{O}_{\Sigma_{12}^t}, \mathcal{O}_{\Sigma_{23}})) \in D^b(\mathcal{C}oh_{G \times G}(X \times X)),$$

where s_{ij} are the projections of $X \times (X//G) \times X$ to pairwise products. $\qquad \Box$

Remark. Note that that the restriction of the projection

$$p_{13} : \Sigma_{12} \cap \Sigma_{23}^t \rightarrow (X//G) \times (X//G)$$

is a finite morphism, so we can replace Rp_{13*} by p_{13*}.

Thus, the functors Φ, Ψ are mutually inverse equivalences of categories if and only if
(a) the kernel $\mathcal{L}$ is quasi-isomorphic to the structure sheaf of the diagonal on $(X//G) \times (X//G)$,
(b) the kernel $\mathcal{M}$ is quasi-isomorphic to the structure sheaf of diagonal $\Delta \subset X \times X$ tensored with the regular representation $\mathbf{R} = \mathbf{C}[G]$ of G.

1.4. Suppose now that X is a smooth surface. Then $X^{[m]}$ is smooth and $X//G$ is an irreducible component of the fixed point set of the G-action on $X^{[m]}$, so it is also smooth. Thus p is a resolution of singularities of X/G. The following theorem is the main result of the section.

Theorem. *Let X be a surface equipped with a holomorphic symplectic form ω, and suppose that the G-action on X preserves ω. Then Φ and Ψ are mutually inverse equivalences of categories.*

From the definitions it is clear that all the cohomology sheaves of $\mathcal{L}$ are supported on the $([\xi], [\eta])$ such that $\mathrm{supp}(\xi) \cap \mathrm{supp}(\eta) \neq \emptyset$, and the cohomology sheaves of $\mathcal{M}$ are supported on $\bigcup_{g \in G}(1 \times g)(\Delta)$. Thus it is enough to work in the neighborhoods of fixed points of subgroups of G, where the action can be replaced by the linear one. We now assume this to be the case.

1.5. Given a finite subgroup $G \subset SL_2(\mathbf{C})$ let τ be the natural representation of G on $\mathbf{C}^2$. For any irreducible representations π, ρ of G let $m_{\pi\rho}$ be the multiplicity of π in $\rho \otimes_{\mathbf{C}} \tau$. We now assume $X = \tau$ and $G \subset SL_2(\mathbf{C})$. Set $\mathbf{A} = \mathbf{C}[x, y]$. Let M be any $\mathbf{A}$-module. Then, we have the (Koszul) free resolution of M by

$$\mathbf{A} \otimes_{\mathbf{C}} M \xrightarrow{(x,y)} (\mathbf{A} \otimes_{\mathbf{C}} M)^{\oplus 2} \xrightarrow{\binom{y}{-x}} \mathbf{A} \otimes_{\mathbf{C}} M.$$

Constructing this resolution simultaneously for $M = \mathcal{O}_\xi$ and all $[\xi] \in X//G$, we get a resolution of $\mathcal{O}_\Sigma$ by the complex on $(X//G) \times X$:

$$\mathcal{K} = \{p_1^*\mathcal{E} \rightarrow (p_1^*\mathcal{E})^{\oplus 2} \rightarrow p_1^*\mathcal{E}\}.$$

Thus
$$\Phi(\mathcal{F}) = (Rp_{1*}(p_2^*\mathcal{F} \otimes^L \mathcal{K}))^G$$
$$= \{(Rp_{1*}p_2^*\mathcal{F}) \otimes \mathcal{E} \to (Rp_{1*}p_2^*\mathcal{F}) \otimes \mathcal{E}^{\oplus 2} \to (Rp_{1*}p_2^*\mathcal{F}) \otimes \mathcal{E}\}^G.$$

Note that $Rp_{1*}p_2^*\mathcal{F}$ is just the trivial bundle on $X//G$ with fiber $\Gamma(X, \mathcal{F})$ (the higher cohomology vanishes on $X = \tau$). Moreover, if $\mathcal{F}$ is a G-equivariant sheaf, then $\Gamma(X, \mathcal{F})$ is a $\mathbf{A}$-module with G-action and can be split $\Gamma(X, \mathcal{F}) = \bigoplus_\pi \pi \otimes_{\mathbf{C}} \Gamma(X, \mathcal{F})_\pi$, so $\Phi(\mathcal{F})$ can be rewritten as

$$\Phi(\mathcal{F}) = \{\bigoplus_\pi \Gamma(X, \mathcal{F})_\pi \otimes_{\mathbf{C}} \mathcal{E}_\pi \to \bigoplus_{\pi,\rho} \Gamma(X, \mathcal{F})_\pi \otimes_{\mathbf{C}} \mathcal{E}_\rho^{m_{\pi\rho}} \to \bigoplus_\pi \Gamma(X, \mathcal{F})_\pi \otimes_{\mathbf{C}} \mathcal{E}_\pi\}.$$

Similarly we get

$$\Psi(\mathcal{G}) = \{R\Gamma(X//G, \mathcal{G} \otimes \mathcal{E}^*) \otimes_{\mathbf{C}} \mathcal{O}_\tau \to R\Gamma(X//G, \mathcal{G} \otimes \mathcal{E}^*) \otimes_{\mathbf{C}} \mathcal{O}_\tau^2 \to R\Gamma(X//G, \mathcal{G} \otimes \mathcal{E}^*) \otimes_{\mathbf{C}} \mathcal{O}_\tau\}.$$

By using the Koszul resolution $\mathcal{K}$, we find that $\mathcal{L}$ is quasi-isomorphic to

$$\mathcal{L}' = (Rp_{13*}(R\mathcal{H}om(p_{12}^*\mathcal{K}, \mathcal{O}_{\Sigma_{23}^t})))^G$$

$$= \{\mathcal{E}^* \underline{\otimes} \mathcal{E} \to (\mathcal{E}^* \underline{\otimes} \mathcal{E})^{\oplus 2} \to \mathcal{E}^* \underline{\otimes} \mathcal{E}\}^G$$

$$= \{\bigoplus_\pi \mathcal{E}_\pi^* \underline{\otimes} \mathcal{E}_\pi \to \bigoplus_{\pi,\rho} \mathcal{E}_\pi^* \underline{\otimes} \mathcal{E}_\rho^{m_{\pi\rho}} \to \bigoplus_\pi \mathcal{E}_\pi^* \underline{\otimes} \mathcal{E}_\pi\},$$

where $\underline{\otimes}$ is the external tensor product. The quasi-isomorphism $\mathcal{L}' \simeq \mathcal{O}_\Delta$ is proved in [11, Lemma 4.10]. By using the Koszul resolution for $\mathcal{O}_{\Sigma_{12}^t}$ we represent $\mathcal{M}$ by the quasi-isomorphic complex

$$\mathcal{M}' = \{\mathcal{O}_\tau \underline{\otimes} Rq_{2*}q_1^*\mathcal{E}^* \to \left(\mathcal{O}_\tau \underline{\otimes} Rq_{2*}q_1^*\mathcal{E}^*\right)^{\oplus 2} \to \mathcal{O}_\tau \underline{\otimes} Rq_{2*}q_1^*\mathcal{E}^*\}.$$

To show that $\mathcal{M}'$ is quasi-isomorphic to $\mathcal{O}_\Delta \otimes_{\mathbf{C}} \mathbf{R}$, it is enough to show that $Rq_{2*}(q_1^*\mathcal{E}^*) = \mathcal{O}_\tau \otimes_{\mathbf{C}} \mathbf{R}$. Then $\mathcal{M}'$ will be identified with the tensor product of $\mathbf{R}$ and the Koszul resolution of the diagonal in $\mathbf{C}^2 \times \mathbf{C}^2$:

$$\mathcal{O} \underline{\otimes} \mathcal{O} \to \mathcal{O} \underline{\otimes} \mathcal{O}^{\oplus 2} \to \mathcal{O} \underline{\otimes} \mathcal{O}.$$

Applying the Koszul resolution one more time, we are reduced to the following fact.

Proposition. *We have* $\Gamma(X//G, \mathcal{E}^* \otimes \mathcal{E}) = \mathbf{R}[x, y]$ *and* $H^i(X//G, \mathcal{E}^* \otimes \mathcal{E}) = 0 \quad \forall i > 0.$

Proof: Recall that $\Gamma(X//G, \mathcal{E}) = \mathbf{A}$ (see [3]) and thus for any $\mathcal{G} \in \mathcal{C}oh(X//G)$ the space $\Gamma(X//G, \mathcal{E} \otimes \mathcal{G})$ is a $\mathbf{A}$-module. We define a morphism of $\mathbf{A}$-modules $u : \mathbf{R}[x, y] \to \Gamma(X//G, \mathcal{E}^* \otimes \mathcal{E})$ by

$$u(g) = g \in \text{Hom}(\mathcal{E}, \mathcal{E}) = \Gamma(X//G, \mathcal{E}^* \otimes \mathcal{E}), \qquad \forall g \in G.$$

Since the G-action on τ is free outside 0, the space $\Gamma(X//G, \mathcal{E}^* \otimes \mathcal{E})$ is a torsion free $\mathbf{A}$-module, and u is an isomorphism outside $0 \in \tau$. Thus the first assertion follows from the following lemma.

Lemma. *Let $u : M \to N$ be a homomorphism of $\mathbf{A}$-modules such that:*
(a) M is free, and N has no torsion;
(b) u is an isomorphism outside a point (in particular, it is injective).
Then u is an isomorphism.

Proof: Let us regard M, N as coherent sheaves $\mathcal{M}, \mathcal{N}$ on $\mathbf{C}^2$, let U be the complement of the point, so that over U the map $u : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Let $j : U \to \mathbf{C}^2$ be the embedding. Then, taking the direct image, we find a homomorphism

$$j_* j^* u : j_* j^* \mathcal{M} \to j_* j^* \mathcal{N}.$$

It is an isomorphism since $j^* u$ is. Since $\mathcal{M}$ is free, $j_* j^* \mathcal{M} = \mathcal{M}$. On the other hand, since $\mathcal{N}$ is torsion free, the natural map $\mathcal{N} \to j_* j^* \mathcal{N}$ is an embedding. But its image contains the image of $j_* j^* u$, so it is an isomorphism. So $u = j_* j^* u$ is an isomorphism, as claimed. $\square$

To prove the second assertion of the proposition it is enough to show that $R^i p_* (\mathcal{E}^* \otimes \mathcal{E}) = 0$ if $i > 0$. The map p is an isomorphism everywhere except for $0 \in X/G$. The proposition follows from the vanishing of $H^i(p^{-1}(0), \mathcal{E}^* \otimes \mathcal{E})$ if $i > 0$, proved in Remark 2.1 below. $\square$

§2. Detailed analysis of the equivalence.

In this section we always assume that $X = \tau$ and G is a finite subgroup in $SL_2(\mathbf{C})$.

2.1. The quotient surface X/G has an isolated singularity at 0. The map p is the minimal desingularization of X/G (see [5]). The vector bundles $\mathcal{E}_\pi$ on $X//G$ were introduced and studied in [3]. Recall that finite subgroups $G \subset SL_2(\mathbf{C})$ are classified by Dynkin diagrams of finite type A-D-E as follows. Let $E = p^{-1}(0)$ and let E_{red} be the reduced variety.

Proposition. *Let Γ^0 be the Dynkin diagram of G. Let Γ be the affine extension of Γ^0.*
(a) Vertices of Γ^0 are in bijection with components of E. Two vertices are joined by an edge if and only if the corresponding components intersect. This intersection is transverse and consists of one point.
(b) Vertices of Γ are in bijection with irreducible representations of G. The vertices corresponding to π and ρ are joined if and only if $m_{\pi,\rho} \neq 0$. $\square$

Let $\mathbf{P}^1_\pi \subset E_{red}$ be the component corresponding to a nontrivial irreducible representation π. Put $d_\pi = \mathrm{rk}\,\mathcal{E}_\pi = \dim \pi$.

Lemma. *The restriction of $\mathcal{E}_\pi$ to $\mathbf{P}^1_\rho$ is trivial (isomorphic to $\mathcal{O}^{d_\pi}$) if $\pi \neq \rho$ and is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}^{d_\pi - 1}$, if $\pi = \rho$.*

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Proof: Consider the (infinite-dimensional) space $\mathbf{A}_\pi$. By construction of $\mathcal{E}_\pi$, we have a surjective map of sheaves on $\mathbf{P}^1_\rho$:

$$\mathbf{A}_\pi \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^1_\rho} \to \mathcal{E}_\pi|_{\mathbf{P}^1_\rho}.$$

This implies that in the splitting $\mathcal{E}_\pi|_{\mathbf{P}^1_\rho} \simeq \bigoplus_i \mathcal{O}(m_i)$ each $m_i \geq 0$. Since the degree is $\sum_i m_i$, our statement follows from the following result of [2] : the degree of the restriction of $\mathcal{E}_\pi$ to $\mathbf{P}^1_\rho$ is 0 if $\pi \neq \rho$ and 1 if $\pi = \rho$. $\qquad\square$

Remark. By Lemma 2.1, we know that for all $\pi \neq \mathbf{C}$ there exist integers a, b, c, such that

$$(\mathcal{E}^* \otimes \mathcal{E})|_{\mathbf{P}^1_\pi} = \mathcal{O}(-1)^{\oplus a} \oplus \mathcal{O}(0)^{\oplus b} \oplus \mathcal{O}(1)^{\oplus c}.$$

Hence $H^i(\mathbf{P}^1_\pi, \mathcal{E}^* \otimes \mathcal{E}) = 0$ for all $i > 0$. The irreducible components E_π of E may be non reduced but have the self intersection number (-2). Therefore, for any vector bundle $\mathcal{F}$ on $X//G$ the restriction $\mathcal{F}|_{E_\pi} = \mathcal{F} \otimes \mathcal{O}_{E_\pi}$ has a filtration with quotients of the form $\mathcal{F}|_{\mathbf{P}^1_\pi} \otimes \mathcal{O}_{\mathbf{P}^1_\pi}(2j), j \geq 0$. Applying this to $\mathcal{F} = \mathcal{E}^* \otimes \mathcal{E}$, we find that its restriction to each component E_π has no higher cohomology. As a consequence, we get

$$H^i(E, \mathcal{E}^* \otimes \mathcal{E}) = 0, \qquad \forall i > 0.$$

2.2. Lemma. *(a) If $[\xi] \in E$ then $\operatorname{Tor}_0(\mathcal{O}_\xi, \mathcal{O}_0) = \mathbf{C}$, $\operatorname{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0) = \mathbf{C} \oplus \bigoplus_{[\xi] \in \mathbf{P}^1_\pi} \pi$ and $\operatorname{Tor}_2(\mathcal{O}_\xi, \mathcal{O}_0) = \bigoplus_{[\xi] \in \mathbf{P}^1_\pi} \pi$.*
(b) The line bundles on $\mathbf{P}^1_\pi$ formed by the $\operatorname{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0)_\pi$ and the $\operatorname{Tor}_0(\mathcal{O}_\xi, \mathcal{O}_0)^G$ are isomorphic to $\mathcal{O}(-1)$ and $\mathcal{O}$ respectivelly.

Proof: Let $\mathbf{m} = (x, y) \subset \mathbf{A}$ and let $\mathbf{n} \subseteq \mathbf{m}$ be the ideal generated by $\mathbf{m}^G$. If $[\xi] \in E$, then the ideal $I_\xi \subset \mathbf{A}$ contains $\mathbf{n}$ (the G-module $\mathcal{O}_\xi = \mathbf{A}/I_\xi$ is isomorphic to the regular representation, in particular, $\dim \mathcal{O}_\xi^G = 1$. If an invariant $f \in \mathbf{n}$ does not lie in I_ξ, then $f \bmod I_\xi \in \mathcal{O}_\xi$ gives a G-invariant on $\mathcal{O}_\xi$ not proportional to 1, so the dimension of $\mathcal{O}_\xi^G$ is at least 2, a contradiction). If $W \subset \mathbf{m}/\mathbf{n}$ is G-invariant set $I(W) = \mathbf{A} \cdot W + \mathbf{n}$.

Theorem [5]. *Let π be a nontrivial irreducible representation of G. Then :*
(a) There exist two different irreducible submodules $\pi', \pi'' \subset \mathbf{m}/\mathbf{n}$, isomorphic to π and such that if $[\xi] \in \mathbf{P}^1_\pi - \bigcup_{\rho \neq \pi} \mathbf{P}^1_\rho$ then $I_\xi = I(W)$, where $W \subset \pi' \oplus \pi''$ is a proper nonzero G-submodule.
(b) If $m_{\pi,\rho} \neq 0$ then the point $[\xi] \in \mathbf{P}^1_\pi \cap \mathbf{P}^1_\rho$ has the ideal $I_\xi = I(\pi' \oplus \rho'')$. $\qquad\square$

Part (a) for Tor_0 and the second claim of part (b) of Lemma follow from the equality $\mathcal{O}_\xi \otimes \mathcal{O}_0 = \mathbf{C}$. Since

$$\operatorname{Tor}_i(\mathcal{O}_\xi, \mathcal{O}_0) = \operatorname{Tor}_{i-1}(I_\xi, \mathcal{O}_0), \quad i = 1, 2,$$

we get $\mathrm{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0) = I_\xi/\mathbf{m}I_\xi$. If $[\xi] \in E$ then $I_\xi = I(W)$ where $W \subset \mathbf{m}/\mathbf{n}$ and $W_\pi = \bigoplus_{[\xi] \in \mathbf{P}^1_\pi} \pi$. Now observe that $W = I_\xi/(\mathbf{m}I_\xi + \mathbf{n})$ and that $((\mathbf{m}I_\xi + \mathbf{n})/\mathbf{m}I_\xi)_\pi = 0$ for any $\pi \neq \mathbf{C}$ since $(\mathbf{m}I_\xi + \mathbf{n})/\mathbf{m}I_\xi \simeq \mathbf{n}/(\mathbf{n} \cap \mathbf{m}I_\xi)$ is a quotient of $\mathbf{n}/\mathbf{mn}$. Thus, if $\pi \neq \mathbf{C}$, $\mathrm{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0)_\pi$ is 1-dimensional for $[\xi] \in \mathbf{P}^1_\pi$ and vanishes otherwise. It remains to study $\mathrm{Tor}_2(\mathcal{O}_\xi, \mathcal{O}_0)_\pi$ and $\mathrm{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0)^G$. The Tor's in question are the cohomology of the Koszul complex

$$\mathcal{O}_\xi \to \mathcal{O}_\xi \otimes_{\mathbf{C}} \tau \to \mathcal{O}_\xi.$$

Since $\mathrm{Tor}_2(\mathcal{O}_\xi, \mathcal{O}_0) \subseteq \mathbf{A}/I_\xi$ and $(\mathbf{A}/I_\xi)^G = \mathbf{C} + I_\xi$, we get $\mathrm{Tor}_2(\mathcal{O}_\xi, \mathcal{O}_0)^G = 0$. Part (a) follows since the equivariant Euler characteristic of the complex is 0 (note that $\mathcal{O}_\xi \otimes_{\mathbf{C}} \tau \simeq \mathcal{O}_\xi \otimes_{\mathbf{C}} \mathbf{C}^2$ since $\mathcal{O}_\xi$ is the regular representation of G). To see part (b), notice that $\mathbf{P}^1_\rho \simeq \mathbf{P}\mathrm{Hom}_G(\rho, \rho' \oplus \rho'')$, and that the bundle formed by the $\mathrm{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0)_\rho = (I_\xi/\mathbf{m}I_\xi)_\rho$ on this $\mathbf{P}^1$ is just $W \mapsto W_\rho$, i.e. $\mathcal{O}(-1)$. $\qquad\square$

2.3. Any finite-dimensional representation V of G gives rise to two equivariant sheaves on X : the skyscraper sheaf $V^!$ whose fiber at 0 is V and all the other fibers vanish, and the locally free sheaf $\tilde{V} = V \otimes_{\mathbf{C}} \mathcal{O}_X$. If π is irreducible, then $\Phi(\tilde{\pi}) = \mathcal{E}_\pi$. The sheaf $\pi^!$ is quasi isomorphic to the (Koszul) complex

$$\tilde{\pi} \otimes_{\mathbf{C}} \Lambda^2 \tau \to \tilde{\pi} \otimes_{\mathbf{C}} \tau \to \tilde{\pi}.$$

Theorem. *We have* $\Phi(\mathbf{C}^!) = \mathcal{O}_E$ *and* $\Phi(\pi^!) = \mathcal{O}_{\mathbf{P}^1_\pi}(-1)[1]$ *if* $\pi \neq \mathbf{C}$.

Proof: Let $\mathcal{F} = \Phi(\pi^!)$. Let us view it as the complex of locally free sheaves on $X//G$ (see (1.5))

$$\mathcal{F} = \{\mathcal{E}_\pi \to \bigoplus_\rho \mathcal{E}_\rho^{m_{\pi\rho}} \to \mathcal{E}_\pi\}.$$

Then for $[\xi] \in X//G$

(2.3.1) $$\mathrm{Tor}_i(\mathcal{O}_\xi, \mathcal{O}_0)_\pi = H^{-i}([\xi], \mathcal{F}_{[\xi]}).$$

Recall that we have a spectral sequence

(2.3.2) $$\mathrm{Tor}_i(\mathcal{H}^{-j}(\mathcal{F}), \mathcal{O}_{[\xi]}) \Rightarrow H^{-i-j}([\xi], \mathcal{F}_{[\xi]}).$$

Lemma 2.2, (2.3.1) and (2.3.2) imply that $\mathcal{H}^0(\mathcal{F}) = 0$ if $\pi \neq \mathbf{C}$. Moreover, if $\pi = \mathbf{C}$, if Λ is any ring and if $[\xi] \in (X//G)(\Lambda)$ is the Λ-point corresponding to the subscheme $\xi \subset X \times \mathrm{Spec}(\Lambda)$ then

$$\mathcal{H}^0(\mathcal{F})_{[\xi]} = \mathrm{Tor}_0(\mathcal{O}_\xi, \mathcal{O}_0)^G = \mathrm{Tor}_0(\mathcal{O}_{\bar\xi}, \mathcal{O}_{\bar 0}) = \mathcal{O}_{E,[\xi]},$$

where $\bar\xi$ and $\bar 0$ are the projection of ξ and 0 in X/G. Thus, $\mathcal{H}^0(\mathcal{F}) = \mathcal{O}_E$.

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By (2.3.1), (2.3.2) and Lemma 2.2, if $\pi \neq \mathbf{C}$ then $\mathcal{H}^{-1}(\mathcal{F})$ is supported on $\mathbf{P}_\pi^1$ and its restriction onto $\mathbf{P}_\pi^1$ is

$$\mathcal{H}^{-1}(\mathcal{F})/I_{\mathbf{P}_\pi^1}\mathcal{H}^{-1}(\mathcal{F}) \quad \simeq \mathcal{O}_{\mathbf{P}_\pi^1}(-1).$$

This does not yet imply that $\mathcal{H}^{-1}(\mathcal{F})$ is actually a sheaf on $\mathbf{P}_\pi^1$ rather than on some infinitesimal neighborhood. For this, we need to show that $\mathcal{H}^{-1}(\mathcal{F})$ is annihilated by the sheaf of ideals $I_{\mathbf{P}_\pi^1}$. Observe that in the category of G-equivariant sheaves we have $\mathrm{End}_G(\pi^!) = \mathbf{C}$, so all the endomorphisms of $\mathcal{F}$ in $D^b(\mathcal{C}oh(X//G))$ are scalar. But $\mathbf{P}_\pi^1$ is a (-2)-curve, so it possesses a lot of functions regular in an entire neighborhood of $\mathbf{P}_\pi^1$ and vanishing on $\mathbf{P}_\pi^1$. So if $\mathcal{H}^{-1}(\mathcal{F})$ actually is not annihilated by $I_{\mathbf{P}_\pi^1}$, there will be a section f of $I_{\mathbf{P}_\pi^1}$ on some neighborhood of $\mathbf{P}_\pi^1$ which is not annihilating $\mathcal{H}^{-1}(\mathcal{F})$. Since all the homology of $\mathcal{F}$ is supported on $\mathbf{P}_\pi^1$, the multiplication by such an f defines an endomorphism of $\mathcal{F}$ in the derived category. This endomorphism is not scalar, since the induced endomorphism on $\mathcal{H}^{-1}(\mathcal{F})$ is not scalar. This contradicts the assumption. So we have established that $\mathcal{H}^{-1}(\mathcal{F}) = \mathcal{O}_{\mathbf{P}_\pi^1}(-1)$ if $\pi \neq \mathbf{C}$. If $\pi = \mathbf{C}$ then $\mathrm{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0)^G = \mathbf{C}$ and $\mathrm{Tor}_1(\mathcal{H}^0(\mathcal{F}), \mathcal{O}_{[\xi]}) = \mathcal{O}(-E)_{[\xi]} = \mathbf{C}$. Since $\mathrm{Tor}_1(\mathcal{O}_\xi, \mathcal{O}_0)^G$ is composed from $\mathrm{Tor}_1(\mathcal{H}^0(\mathcal{F}), \mathcal{O}_{[\xi]})$ and $\mathrm{Tor}_0(\mathcal{H}^{-1}(\mathcal{F}), \mathcal{O}_{[\xi]})$ we have $\mathcal{H}^{-1}(\mathcal{F}) = 0$.

Finally, $\mathcal{H}^{-2}(\mathcal{F}) = 0$. More precisely, if $\pi \neq \mathbf{C}$ then $\mathrm{Tor}_2(\mathcal{O}_\xi, \mathcal{O}_0)_\pi = \mathbf{C}$ is composed from $\mathrm{Tor}_1(\mathcal{H}^{-1}(\mathcal{F}), \mathcal{O}_{[\xi]})$ (which is nonzero) and $\mathcal{H}^{-2}(\mathcal{F}) \otimes \mathcal{O}_{[\xi]}$. If $\pi = \mathbf{C}$ then $\mathrm{Tor}_2(\mathcal{O}_\xi, \mathcal{O}_0)^G = 0$. $\qquad \square$

§3. Hall algebras and double quivers.

3.1. Let us describe a version of the Hall algebra construction [12] based on Euler characteristic. Let $\mathcal{A}$ be a $\mathbf{C}$-linear Abelian category of finite type (i.e. the extension groups between pairs of objects in $\mathcal{A}$ are finite dimensional). If A, B, C are three objects of $\mathcal{A}$, the set $G_{AB}^C = \{A' \subseteq C : A' \simeq A, C/A' \simeq B\}$ has the structure of a complex variety. To see this, let Com_{AB}^C and E_{AB}^C be the set of all complexes and all exact sequences respectively of the form $0 \to A \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$. Clearly, Com_{AB}^C is a closed algebraic subvariety in the affine space $\mathrm{Hom}(A, C) \oplus \mathrm{Hom}(C, B)$, and E_{AB}^C is a Zariski open subset in Com_{AB}^C. Now, the algebraic group $\mathrm{Aut}(A) \times \mathrm{Aut}(C)$ acts on E_{AB}^C freely. The quotient is therefore equipped with a structure of a complex variety. But as a set, this quotient is nothing but G_{AB}^C. Since the heart of a triangulated category is stable by extensions, we get

Proposition. *Let $\mathcal{A}, \mathcal{B}$ be two Abelian categories as above, and $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ be an equivalence of triangulated categories. If A, B, C are objects of $\mathcal{A}$ such that $F(A), F(B) \in \mathcal{B}$ and if $G_{AB}^C \neq \emptyset$, then $F(C) \in \mathcal{B}$ and F is an isomorphism of complex varieties $G_{AB}^C \to G_{F(A),F(B)}^{F(C)}$.* $\qquad \square$

3.2. The characteristic function of a closed subvariety W of an algebraic variety Z is denoted by $\mathbf{1}_W$. By $\mathrm{Fun}(Z)$ we denote the space of all constructible functions on Z. If

$f \in \mathrm{Fun}(Z)$ its integral is:

$$\int_Z f d\chi = \sum c_i \chi(W_i), \quad f = \sum c_i \mathbf{1}_{W_i}.$$

Here χ is the Euler characteristic with compact support. More generally, if $\varphi : Z_1 \to Z_2$ is any regular map and $f \in \mathrm{Fun}(Z_1)$, then its direct image $\varphi_*(f) \in \mathrm{Fun}(Z_2)$ is (cf.[2]):

$$(\varphi_* f)(z_2) = \int_{\phi^{-1}(z_2)} f d\chi.$$

Similarly let $\mathcal{S}$ be an algebraic stacks of locally finite type. The set of $\mathbf{C}$-points $\mathcal{S}(\mathbf{C})$ is locally represented as the quotient of an algebraic variety by an action of an algebraic group. A constructible function on a stack $\mathcal{S}$ is a function $\mathcal{S}(\mathbf{C}) \to \mathbf{C}$ which can be represented as a finite linear combination of functions of the form $\mathbf{1}_{\mathcal{W}(\mathbf{C})}$, where $\mathcal{W}$ is a closed substack of finite type in $\mathcal{S}$. Let $\varphi : \mathcal{S} \to \mathcal{T}$ be a morphism of stacks whose every fiber (over any $\mathbf{C}$-point) is an algebraic variety. Then we have the direct image map $\varphi_* : \mathrm{Fun}(\mathcal{S}) \to \mathrm{Fun}(\mathcal{T})$ defined as above.

3.3. We now assume that $\mathcal{A}$ comes from a stack of Abelian categories over the category of algebraic varieties, and that the moduli stack of objects of $\mathcal{A}$, denoted by $\mathcal{A}^{iso}$, is an algebraic stack of locally finite type.

Examples. (1) $\mathcal{A}$ is the category of representations of a finite-dimensional $\mathbf{C}$-algebra. (2) $\mathcal{A}$ is the category of coherent sheaves on an algebraic variety with support in a projective subvariety.

The set $\mathcal{A}^{iso}(\mathbf{C})$ is the set of isomorphism classes of objects of $\mathcal{A}$. The space $\mathrm{Fun}(\mathcal{A}^{iso})$ is made into an associative algebra, called the Hall algebra of $\mathcal{A}$ and denoted by $\mathbf{H}(\mathcal{A})$, as follows. Let $\mathcal{G}_\mathcal{A}$ be the stack formed by pairs (A, B) of objects of $\mathcal{A}$, with A a subobject of B, and morphisms of such pairs. There are three morphisms $p_1, p_2, p_3 : \mathcal{G}_\mathcal{A} \to \mathcal{A}^{iso}$ which associate to (A, B) the objects A, B, and B/A respectively. The fibers of p_2 are algebraic varieties. The multiplication on $\mathbf{H}(\mathcal{A})$ is

$$f * g = p_{2*}((p_1^* f) \cdot (p_3^* g)).$$

Let $[A] \in \mathbf{H}(\mathcal{A})$ be the characteristic function of the object A. Then $\chi(G_{AB}^C)$ is the multiplicity of $[C]$ in $[A] * [B]$.

3.4. Let Γ be any finite graph without loops and multiple edges. A double representation of Γ is a rule which assigns to each vertex i a vector space V_i, and to any edge $\{i, j\}$ two operators $x_{ij} : V_j \to V_i$ and $x_{ji} : V_i \to V_j$ such that for every vertex i we have $\sum_j x_{ij} x_{ji} = 0$. Finite dimensional double representations of Γ form an Abelian category of finite type and global dimension 2 if Γ is of affine type, denoted $\mathcal{R}_\Gamma$.

Proposition. *Let Γ be an affine Dynkin graph of type A-D-E, corresponding to a finite subgroup $G \subset SL_2(\mathbf{C})$. The category $\mathcal{R}_\Gamma$ is equivalent to $\mathcal{C}oh_G(\tau)$.*

Proof: Let $\{\pi_i\}_{i \in I}$ be the set of simple representations of G and let $J = \{(i,j) \,|\, \pi_j \subset \pi_i \otimes \tau\}$ be the set of edges of the graph Γ. A G-equivariant coherent sheaf on τ is a pair (V, ϕ) where V is a finite dimensional G-module and ϕ is a G-invariant linear map $V \otimes \mathbf{C}[\tau] \to V$. Such a pair (V, ϕ) may be viewed as

    (i) a I-graded vector space $\bigoplus_{i \in I} V_{\pi_i}$,

    (ii) a collection of maps $\bigoplus_{(i,j) \in J} \,:\, V_{\pi_i} \otimes \tau \to \bigoplus_{(i,j) \in J} V_{\pi_j}$.

Then the relation $\sum_j x_{ij} x_{ji} = 0$ means precisely that the maps in (ii) glue together in a map $V \otimes \mathbf{C}[\tau] \to V$. $\qquad\square$

Let $\mathbf{C}(i)$ be the simple object in $\mathcal{R}_\Gamma$ located at the vertex i of Γ. Put $\theta_i = [\mathbf{C}(i)] \in \mathbf{H}(\mathcal{R}_\Gamma)$. As usual, set $\theta_i^{(k)} = \theta_i^k / k!$ for any $k \in \mathbf{N}^\times$. Denote by A_Γ the Cartan matrix of Γ, such that the index set is the set of vertices of Γ, $a_{ii} = 2$, and $-a_{ij}$ is the number of edges joining i and j if $i \neq j$. Let $\mathbf{g}_\Gamma$ be the Kac-Moody Lie algebra associated to A_Γ. The "nilpotent" subalgebra $\mathbf{g}_\Gamma^+ \subset \mathbf{g}_\Gamma$ is generated by the Chevalley generators $\mathbf{e}_i$ subject to the Serre relations

$$\forall i \neq j, \quad \sum_k (-1)^k \mathbf{e}_i^{\pm\,(k)} \mathbf{e}_j^\pm \mathbf{e}_i^{\pm\,(1 - a_{ij} - k)} = 0.$$

The following is a reformulation of a result from [8].

Theorem. *The correspondence $\mathbf{e}_i \to \theta_i$ defines a homomorphism $\mathbf{U}(\mathbf{g}_\Gamma^+) \to \mathbf{H}(\mathcal{R}_\Gamma)$.* $\qquad\square$

§4. Applications to algebraic surfaces.

Let S be a smooth surface over $\mathbf{C}$ and $C \subset S$ be a reducible curve of which every component is a rational reduced (-2)-curve, such that these curves meet transversely and not more than in one point. Let $\mathbf{P}_i^1, i \in I$, be the irreducible components of C. Let A_C be the negative of the intersection matrix of the components of C and let $\mathbf{g}_C$ be the Kac-Moody Lie algebra with Cartan matrix A_C. Denote by $\mathcal{C}oh(S, C)$ the category of coherent sheaves on S with support in C. This category has finite type. Let $\mathbf{H}(S, C)$ be its Hall algebra.

Theorem. *The correspondence $\mathbf{e}_i \mapsto [\mathcal{O}_{\mathbf{P}_i^1}]$ defines an algebra homomorphism $\mathbf{U}(\mathbf{g}_C^+) \to \mathbf{H}(S, C)$.*

Corollary. *If S is a projective surface then $\mathbf{U}(\mathbf{g}_C^+)$ acts on $\mathrm{Fun}(\mathcal{C}oh(S))$.* $\qquad\square$

Proof: It is enough to consider the case when C has type A_2. It is well-known (see [1], Theorem 7.3) that if C is a configuration of (-2)-curves on S such that A_C has type A-D-E then the formal neighborhood of C in S is isomorphic to the formal neighborhood of the exceptional fiber E in the Hilbert quotient $\mathbf{C}^2 /\!/ G$ where $G \subset SL_2(\mathbf{C})$ is the finite

subgroup with Cartan matrix A_C. Then we have the equivalence of categories $\mathcal{C}oh(S, C) \simeq \mathcal{C}oh(\mathbf{C}^2//G, E)$. Now, Theorem 1.4 and Propositions 3.1 and 3.4 imply that the subalgebra in $\mathbf{H}(\mathcal{R}_\Gamma)$ generated by the θ_π, $\pi \neq \mathbf{C}$, is isomorphic to the subalgebra in $\mathbf{H}(\mathbf{C}^2//G, E)$ generated by the $[\mathcal{O}_{\mathbf{P}^1_\pi}]$. Thus our theorem follows from Theorem 3.4. $\qquad \square$

References

[1] W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces, *Springer* (1984).

[2] W. Fulton, R. MacPherson, Categorical Framework for the Study of Singular Spaces, *Memoirs AMS* **243**, 1981.

[3] G. Gonzalez-Sprinberg, J.L. Verdier, Construction géométrique de la correspondance de McKay, *Ann. ENS*, **16** (1983), 409-449.

[5] Y. Ito, I. Nakamura, Hilbert schemes and simple singularities, *New trends in algebraic geometry (Proceedings of the algebraic symposium, Warwick 1996)*, Cambridge University Press (1999), 151-233.

[6] M. Kapranov, Chow quotients of Grassmannians, in: "I.M. Gelfand Seminar" (S. Gelfand, S. Gindikin, Eds.), vol.1, (*Adv. in Soviet Math.*, v. 16, pt.2), p. 29-110, Amer. Math. Soc. Providence RI 1993.

[7] P.B. Kronheimer, H. Nakajima, Yang-Mills instantons on ALE gravitational instantons, *Math. Ann.* **288** (1990), 263-307

[8] G. Lusztig, Quivers, perverse sheaves and enveloping algebras, *J. Amer. Math. Soc.*, **4** (1991), 365-421.

[9] J. McKay, Graphs, singularities and finite groups, in: "The Santa Cruz conference on finite groups' (Santa Cruz CA 1979), p. 183-186, Pro. Symp. Pure Math. **37**, Amer. Math. Soc. Providence RI 1980.

[10] H. Nakajima, Instantons on ALE spaces, quiver varieties and Kac-Moody algebras, *Duke Math. J.* **76** (1994), 365-416.

[11] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, *Preprint* (1998).

[12] C.M. Ringel, Hall algebras and quantum groups, *Invent. Math.* **101** (1990), 583-592.

M.K. : Department of Mathematics, Northwestern University,
Evanston, Illinois 60208. kapranov at math.nwu.edu.

E.V. : Département de Mathématiques, Université de Cergy-Pontoise,
2 Av. A. Chauvin, 95302 Cergy-Pontoise Cedex, France. vasserot at math.pst.u-cergy.fr.